

Characterization of Smart-proof curves

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Abstract

The points of an Elliptic curve over a finite field forms an finite abelian group, hence frequently used in cryptography due to the conjectured difficulty of solving the discrete logarithm problem. However certain classes of curves have computationally simple solutions to the discrete logarithm, for instance curves of trace 1, known as anomalous curves. This attack was first published by Smart, hence its nickname, the ‘Smart Attack’. This attack lifts curves from \mathbb{F}_p to \mathbb{Q}_p . However, it has a small chance of lifting to a curve where the attack fails. This paper’s main objective is to classify such lifts.

1 Introduction

Suppose $kP = Q$ with P, Q known and k unknown. This is the discrete logarithm problem(DLP) for elliptic curves and is generally difficult. However if the trace of the curve is 1, then this can be translated to the DLP over \mathbb{F}_p^+ , which is simply solving $\frac{a}{b} \pmod{p}$. Such curves are known as anomalous curve. From now all curves are assumed to be anomalous.

For a curve E/\mathbb{F}_p , to translate the DLP to \mathbb{F}_p^+ , first lift it to E/\mathbb{Q}_p and define the subgroups of the group of points on E/\mathbb{Q}_p :

$$E_r = \{(x, y) \in E/\mathbb{Q}_p \mid v_p(x) \leq -2r, v_p(y) \leq -3r\} \cup \{\infty\}$$

Note that $\frac{E_0}{E_1} \cong E/\mathbb{F}_p$ and $\frac{E_1}{E_2} \cong \mathbb{F}_p^+$, which the first isomorphism given by reduction mod p last isomorphism given by $\psi : (x, y) \rightarrow -\frac{x}{py}$.

Assume that $kP = Q$ in E/\mathbb{F}_p . Now working in E/\mathbb{Q}_p , we have $pP, pQ, kP - Q \in E_1$ since curve is of order p . $k\psi(pP) - \psi(pQ) = p\psi((kP - Q)) = 0$, so $k = \frac{\psi(pQ)}{\psi(pP)}$, which is a DLP over \mathbb{F}_p^+ and is computationally extremely easy. Note that if $pP \in E_2$, then we get $k = \frac{0}{0}$, so the proof does not work for this case. For such a lift, the lifted curve is called *Smart-proof*

Main Objectives The main objectives of this paper are:

- Show lifts that are Smart-proof occur at a $\frac{1}{p}$ probability

- Show that when $a = kp$, the curve is Smart-proof iff $k = 0 \pmod{p}$
- Find all Smart-proof curves with $0 \leq a, b < p$
- Find all Smart-proof curves

2 Experimental results

For a curve $y^2 = x^3 + ax + b$ over \mathbb{F}_p , $0 \leq a, b < p$, suppose $y^2 = x^3 + (a + mp)x + (b + np)$ over \mathbb{Q}_p is Smart-proof. It can be experimentally shown that:

- For Smart-proof curves, the attack acts like a random number generator except when $P = \pm Q$ where it gives accurate results.
- When $a = 0$, the curve is Smart-proof iff $m = 0$.
- When $a \neq 0$, every value of m has a unique value of n .
- $n(m) = n(0) + km$ for some k coprime to p (treating n as a function of m)
- For some values of p (i.e. 23, 29), k takes on every value once.

3 Smart-proof lifts

Let $f \in \text{End}(E/\overline{\mathbb{F}_p})$ be the Frobenius endomorphism and let \hat{f} be the dual isogeny. The kernel of the dual isogeny is E/\mathbb{F}_p , which has order p , hence it is separable. Suppose that f, \hat{f} gets lifted to an endomorphism $\tilde{f}, \hat{\tilde{f}} \in \text{End}(E/\mathbb{Q}_p)$. Since \hat{f} is separable, $\ker \hat{f} \cong \ker \hat{\tilde{f}} \cong E/\mathbb{F}_p$, which is precisely $E_0[p]$. Then E_0 splits to $E_1 \times E_0[p]$, hence $pE_0 = pE_1$. Conversely, if the kernel is trivial then $pE_0 = E_1$.

These results show that if for any lifted point P , $pP \in E_2$, then this is true for all points, similarly if $pP \in E_1$, then this is true for all points. The former case only occurs iff the Frobenius endomorphism can be lifted to an endomorphism. One way to look at the curve $y^2 = x^3 + ax + b$ over \mathbb{Q}_p is transforming it to \mathbb{Z}_p via $[x : y : z] \rightarrow [x : z : y]$ and we end up with $x^3 + axy^2 + by^3 - y = 0$. Reduce mod p^2 and we have a (abelian) group of points on a curve with p^2 elements, so the group is either $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^2$ or $\frac{\mathbb{Z}}{p^2\mathbb{Z}}$. The former gives a Smart-proof lift while latter gives a non-Smart-proof lift as the only p -torsion points are of the form $[kp : 0 : 1]$.

This transformation also shows that if a, b results in a Smart-proof curve, then after adding p^2 to either a, b , the curve remains Smart-proof. Hence to show that $\frac{1}{p}$ lifts are smart proof, we only need to show that out of the p^2 possible lifts mod p^2 , p of them are Smart-proof.

The addition laws can be derived easily:

$$x^3 + axy^2 + by^3 - y = 0 \quad (3x^2 + ay^2) dx + (2axy + 3by^2 - 1) dy = 0$$

$$\lambda = \begin{cases} \frac{P_y - Q_y}{P_x - Q_x} & P \neq Q \\ \frac{3P_x^2 + aP_y^2}{1 - 2aP_xP_y - 3bP_y^2} & P = Q \end{cases}$$

$$y = \lambda x + (P_y - \lambda P_x)$$

$$P = (P_x, P_y), Q = (Q_x, Q_y), R = P + Q$$

$$x^3 + ax(\lambda x + (P_y - \lambda P_x))^2 + b(\lambda x + (P_y - \lambda P_x))^3 - (\lambda x + (P_y - \lambda P_x)) = 0$$

$$(1 + a\lambda^2 + b\lambda^3)x^3 + \lambda(2a + 3b\lambda)(P_y - \lambda P_x)x^2 + O(x) = 0$$

$$-R_x = \frac{\lambda(2a + 3b\lambda)(\lambda P_x - P_y)}{1 + a\lambda^2 + b\lambda^3} - P_x - Q_x$$

$$-R_y = \lambda R_x + (P_y - \lambda P_x)$$

$$R_x = \frac{\lambda(2a + 3b\lambda)(P_y - \lambda P_x)}{1 + a\lambda^2 + b\lambda^3} + P_x + Q_x$$

$$R_y = -\lambda R_x + (\lambda P_x - P_y)$$

Note if $P_x = Q_x$ and $P_y \neq Q_y$, then $R_x = -P_x$ and $R_y = \frac{aP_x}{b} + P_y + Q_y$